Technical Notes

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Mass and Stiffness Interaction Effects in Analytical Model Modification

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Introduction

STRUCTURAL dynamic analytical models of complex structures require analyses which have frequency responses as important design considerations. The acceptance of an analytical model is often based on a subjective evaluation of the predicted and measured natural frequencies and normal mode shapes. The test data and modeling accuracies are both important considerations in the identification process. No generally accepted method is presently employed which uses the test data objectively to improve the analytical model.

The information obtained from ground vibration tests often leads to the correction of major deficiencies in the modeling of certain areas of the structure. The judgment required in the analytical model improvement is an engineering evaluation of the predicted minimum changes which will exactly predict the measured data.

Berman and Flannelly[1] developed a method for identifying parameters in a linear, discrete model of a structure by using measured normal modes to modify an analytically derived model. This method does not provide for the correction of the analytical stiffness matrix, but synthesizes an incomplete stiffness matrix.

Baruch and Bar-Itzhack[2] assumed that the mass matrix is correct and introduced a constrained minimization procedure to adjust analytical stiffness and flexibility matrices. The resulting analytical dynamic model modes are identical to the analytically orthogonalized test modes used in the identification.

Berman[3] assumed that the test mode shapes are exact and modifies the mass matrix using a minimum-weighted Euclidean norm and Lagrangian multiplier techniques to satisfy the orthogonality constraint.

Kabe[4] introduced a procedure that uses the mode data and structural connectivity information to optimally adjust deficient stiffness matrices. The adjustments performed are such that the percentage change to each stiffness coefficient is minimized.

The comparison of dynamic analysis and dynamic test data obtained from a linear structure rarely demonstrates complete or acceptable compatibility. One simple approach is to first modify the analytical mass matrix to satisfy the orthogonality condition based on the measured mode shapes. Then, the stiffness matrix is modified to satisfy the eigenvalue equation as a function of the measured mode shapes, natural frequencies, and the corrected mass matrix.[5-14]

In Ref. 15, an alternate approach is offered to correct the stiffness matrix using an incomplete set of static loads and deflections obtained from static tests. Then, the analytical mass matrix is modified to fulfill the eigenvalue equation based on the modal test data and the updated stiffness matrix.

All previously published methods have the merits of improving the analytical structural dynamic model; however, the interaction between the mass and stiffness matrices are not taken into consideration in deriving the analytical model. In this Technical Note, both the analytical mass and stiffness matrices are modified simultaneously using the vibration test data. The dynamic model improvement is obtained by minimizing the weighted Euclidean norm from the original analytical model combined with the Lagrange multiplier technique.

The dynamic equation and the orthogonality constraints are satisfied during the analytical derivation. The matrix inversions which are required in the analysis are of the order of the number of measured modes. The effects due to mass and stiffness interaction are determined from the final equations. These interaction effects can be very important to many analytical engineers.

Theoretical Development

Dynamic response is an important consideration in aerospace structure design. Analytical models are used to identify the parameters from modal test results. These analyses use two basic theoretical relationships which apply to linear, undamped structures represented by finite-element models. These are orthogonality of the normal modes, and the eigenvalue equation as described in Eqs. (1) and (2):

$$\Phi^T M \Phi = I \tag{1}$$

$$K\Phi = M\Phi\Omega^2 \tag{2}$$

The measured modes obtained in the ground vibration tests are often fewer than the number of degrees of freedom. The normal modal matrix $\Phi(n \times m)$ is rectangular, where $n \geq m$, and the natural frequency matrix $\Omega^2(m \times m)$ is diagonal. There are an infinite number of mass and stiffness matrices which satisfy the orthogonality and eigenvalue equations. If the changes in the mass and stiffness matrices are minimized, an analytical solution will result under the given weighting matrix condition.

The normalized mode shapes and the measured frequencies obtained from the tests are assumed exact and have to satisfy the basic dynamic relationships of Eqs. (1) and (2). The corrected mass $M(n \times n)$ and stiffness $K(n \times n)$ matrices are symmetric and represent the desired matrices obtained from the analytical model improvement.

The analytical mass matrix $M_A(n \times n)$ is a positive-definite, symmetric matrix, and the analytical stiffness matrix $K_A(n \times n)$ is either a positive-definite or a semidefinite symmetric matrix if it includes rigid-body motions. Both M_A and K_A are assumed to be known matrices.

The convenient way to find the desired mass matrix M and stiffness matrix K which give minimum changes to the known

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analytical matrices M_A and K_A is to minimize the functions ϵ_1 and ϵ_2 defined in Eqs. (3) and (4):

$$\epsilon_1 = \frac{1}{2}\left\|M_A^{-1/2}(K - K_A)M_A^{-1/2}\right\| \tag{3}$$

$$\epsilon_2 = \frac{1}{2}\left\|M_A^{-1/2}(M - M_A)M_A^{-1/2}\right\| \tag{4}$$

Defining a Lagrangian multiplier matrix $(\Delta,\Lambda,\alpha,\beta)$ for each of the constraint equations results in the Lagrangian function Ψ:

$$\Psi = \epsilon_1 + \epsilon_2 + 2\sum_{i=1}^{n}\sum_{j=1}^{m}\Lambda_{ij}(K\Phi - M\Phi\Omega^2)_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{m}\Delta_{ij}(\Phi^T M\Phi - I)_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}(M - M^T)_{ij} + \sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}(K - K^T)_{ij} \tag{5}$$

Here Λ is a rectangular matrix of the order $(n \times m)$ and Δ is a symmetric matrix of order $(m \times m)$. Both α and β are antisymmetric matrices of order $(n \times n)$.

Differentiating Eq. (5) with respect to the elements m_{ij} and k_{ij} and setting these results to zero yields

$$M_A^{-1}(M - M_A)M_A^{-1} - 2\Lambda\Omega^2\Phi^T + \Phi\Delta\Phi^T + 2\alpha = 0 \tag{6}$$

$$M_A^{-1}(K - K_A)M_A^{-1} + 2\Lambda\Phi^T + 2\beta = 0 \tag{7}$$

Adding Eqs. (6) and (7) and their respective transposes eliminates the Lagrangian multipliers α and β:

$$M = M_A + M_A\Lambda\Omega^2\Phi^T M_A + M_A\Phi\Omega^2\Lambda^T M_A - M_A\Phi\Delta\Phi^T M_A \tag{8}$$

$$K = K_A - M_A\Lambda\Phi^T M_A - M_A\Phi\Lambda^T M_A \tag{9}$$

Substituting Eq. (8) into Eq. (1) to eliminate Lagrangian multiplier Δ results in

$$M = M_A + M_A\Lambda Y + Y^T\Lambda^T M_A - M_0 - Z\Lambda Y - Y^T\Lambda^T Z \tag{10}$$

where

$$Y = \Omega^2\Phi^T M_A \tag{11}$$

$$M_0 = M_A\Phi Q^{-1}(Q - I)Q^{-1}\Phi^T M_A \tag{12}$$

$$Z = M_A\Phi Q^{-1}\Phi^T M_A \tag{13}$$

$$Q = \Phi^T M_A\Phi \tag{14}$$

Furthermore, substituting Eqs. (9) and (10) into Eq. (2) and rearranging the equation gives

$$M_A\Lambda = S - P\Omega^2 E^{-1} - M_A\Phi\Lambda^T M_A\Phi E^{-1} + Z\Lambda\Omega^2 Q\Omega^2 E^{-1} \tag{15}$$

where

$$E = Q + \Omega^2 Q\Omega^2 \tag{16}$$

$$P = M_A\Phi Q^{-1} \tag{17}$$

$$S = K_A\Phi E^{-1} \tag{18}$$

Equation (15) and its transpose are substituted into Eq. (10) and the Lagrangian multiplier Λ cancels. The desired mass matrix which has the stiffness matrix correction effects in the last two terms is

$$M = M_A - M_0 + (I - P\Phi^T)SY + Y^T S^T(I - \Phi P^T) \tag{19}$$

Equation (19) is an easily evaluated expression for the corrected mass matrix to make it consistent with the measured data. If the K_A terms are neglected from the equation, the corrected mass matrix is reduced to the results shown in Refs. 6 and 10. The only inversions required are those of Q and E, which are of the order of the number of measured modes.

In order to determine the corrected stiffness matrix which satisfies the eigenvalue equation, Eqs. (9) and (19) are substituted into Eq. (2) to get

$$M_A\Lambda = K_A\Phi Q^{-1} - M_A\Phi\Lambda^T P - S\Omega^2 Q\Omega^2 Q^{-1} - P\Omega^2 Q^{-1} + P\Phi^T S\Omega^2 Q\Omega^2 Q^{-1} \tag{20}$$

Substituting Eq. (20) and its transpose into Eq. (9) gives

$$K = K_0 + 2P\Omega^2 P^T - U\Phi P^T - P\Phi^T U^T + M_A\Phi\Lambda^T Z + Z\Lambda\Phi^T M_A \tag{21}$$

where

$$K_0 = K_A - K_A\Phi P^T - P\Phi^T K_A + U^T + U \tag{22}$$

$$U = P\Omega^2 Q\Omega^2 S^T \tag{23}$$

Again, substituting Eq. (20) and its transpose into Eq. (21) and adding the resulting equation on Eq. (21) yields the desired stiffness matrix:

$$K = K_0 + P(\Phi^T K_A\Phi + \Omega^2)P^T - U\Phi P^T - P\Phi^T U^T \tag{24}$$

Therefore, from Eqs. (19) and (24), the corrected mass and stiffness matrices satisfying both the orthogonality requirement and the eigenvalue equation are developed. The interaction between the mass and stiffness matrices are easily identified from the equations as compared to previously published methods.

Conclusions

The mass and stiffness matrices can be optimally corrected simultaneously from incomplete modal test data by using the Lagrange multiplier method. The dynamic equation and the orthogonality constraints are satisfied during the analytical derivation. The minimized function is formulated such that the resulting changes to the model are a minimum under the given weighting matrix condition. The desired model exactly reproduces the modal and frequency matrices used in the identification. No iteration is needed in order to ensure that the desired matrices satisfy all of the constraint equations. This method appears to have potential as an important tool for studies of the effects of structural changes.

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